J Glob Optim (2007) 39:441–457
DOI 10.1007/s10898-007-9148-y

ORIGINAL ARTICLE

The existence and uniqueness of solution and the convergence of a multi-step iterative algorithm for a system of variational inclusions with (A, η, m)-accretive operators

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Received: 17 August 2006 / Accepted: 8 February 2007 / Published online: 13 April 2007

Abstract In this paper, we introduce and study a new system of variational inclusions with (A, η, m)-accretive operators which contains variational inequalities, variational inclusions, systems of variational inequalities and systems of variational inclusions in the literature as special cases. By using the resolvent technique for the (A, η, m)-accretive operators, we prove the existence and uniqueness of solution and the convergence of a new multi-step iterative algorithm for this system of variational inclusions in real q-uniformly smooth Banach spaces. The results in this paper unifies, extends and improves some known results in the literature.

Keywords System of variational inclusions · (A, η, m)-accretive operator · Existence · Multi-step iterative algorithm · Convergence

AMS Subject Classifications 49J40 · 47J20

1 Introduction

Variational inclusion problems are among the most interesting and intensively studied classes of mathematical problems and have wide applications in the fields of optimization and control, economics and transportation equilibrium, engineering science. For the past years, many existence results and iterative algorithms for various variational inequality and variational inclusion problems have been studied. For details, please see Refs. [1–48] and the references therein.

Recently, some new and interesting problems, which are called to be system of variational inequality problems were introduced and studied. Pang [27], Cohen and Chaplais [28], Bianchi [29] and Ansari and Yao [15] considered a system

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of scalar variational inequalities. And Pang showed that the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium, and the general equilibrium programming problem can be modeled as a system of variational inequalities. Ansari et al. [30] introduced and studied a system of vector equilibrium problems and a system of vector variational inequalities by a fixed point theorem. Allevi et al. [31] considered a system of generalized vector variational inequalities and established some existence results with relative pseudomonotonicity. Kassay and Kolumbán [16] introduced a system of variational inequalities and proved an existence theorem by the Ky Fan lemma. Kassay et al. [17] studied Minty and Stampacchia variational inequality systems with the help of the Kakutani–Fan–Glicksberg fixed point theorem. Peng [18,19] introduced a system of quasi-variational inequality problems and proved its existence theorem by maximal element theorems. Verma [20–24] introduced and studied some systems of variational inequalities and developed some iterative algorithms for approximating the solutions of system of variational inequalities in Hilbert spaces. Kim and Kim [25] introduced a new system of generalized nonlinear quasi-variational inequalities and obtained some existence and uniqueness results of solution for this system of generalized nonlinear quasi-variational inequalities in Hilbert spaces. Cho et al. [26] introduced and studied a new system of nonlinear variational inequalities in Hilbert spaces. They proved some existence and uniqueness theorems of solutions for the system of nonlinear variational inequalities.

As generalizations of above systems of variational inequalities, Agarwal et al. [32] introduced a system of generalized nonlinear mixed quasi-variational inclusions and investigated the sensitivity analysis of solutions for this system of generalized nonlinear mixed quasi-variational inclusions in Hilbert spaces. Kazmi and Bhat [33] introduced a system of nonlinear variational-like inclusions and gave an iterative algorithm for finding its approximate solution. Fang and Huang [34], Verma [35] and Fang et al. [36] introduced and studied a new system of variational inclusions involving H-monotone operators, A-monotone operators and (H, η)-monotone operators, respectively.

On the other hand, Lan et al. [3] and Lan [4] introduced and studied a new concept of (A, η, m)-accretive operators which provides a unifying framework for maximal η-monotone operators in Ref. [5], A-monotone operators in Ref. [35], H-monotone operators in Ref. [1], (H, η)-monotone operators in Ref. [34], (A, η)-monotone operators in Ref. [7], generalized m-accretive operators in Ref. [8], H-accretive operators in Ref. [9], (P, η)-accretive operators in Ref. [10], m-accretive operators in Ref. [12] and maximal monotone operators [13].

Inspired and motivated by the above results, the purpose of this paper is to introduce a new mathematical model, which is called to be a system of variational inclusions with (A, η, m)-accretive operators, i.e. a family of variational inclusions with (A, η, m)-accretive operators defined on a product set. This new mathematical model contains the system of inequalities in Refs. [15,20–29] and the system of inclusions in Refs. [34–36], the variational inclusions in Refs. [1,2,9,11] and some variational inequalities in the literature as special cases. By using the resolvent technique for the (A, η, m)-accretive operators, we prove the existence of solutions for this system of variational inclusions. We also prove the convergence of a multi-step iterative algorithm approximating the solution for this system of variational inclusions. The result in this paper unifies, extends and improves some results in Refs. [1,2,9,11,20–29,34–36].

2 Preliminaries

We suppose that E is a real Banach space with dual space, norm and the generalized dual pair denoted by E^*, $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively, CB$(E)$ is the families of all nonempty closed bounded subsets of E, and the generalized duality mapping $J_q\colon E \to 2^{E^*}$ is defined by

$$J_q(x) = \{f^* \in E^*\colon \langle x, f^*\rangle = \|f^*\| \cdot \|x\|, \|f^*\| = \|x\|^{q-1}\}, \quad \forall x \in E,$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is known that, in general, $J_q(x) = \|x\|^{q-2}J_2(x)$, for all $x \neq 0$, and J_q is single-valued if E^* is strictly convex.

The modulus of smoothness of E is the function $\rho_E\colon [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(t) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1\colon \|x\| \leq 1, \|y\| \leq t\}.$$

A Banach space E is called uniformly smooth if

$$\lim_{t\to 0} \frac{\rho_E(t)}{t} = 0.$$

E is called q-uniformly smooth if there exists a constant $c > 0$, such that

$$\rho_E(t) \leq ct^q, q > 1.$$

Note that J_q is single-valued if E is uniformly smooth. Xu and Roach [49] proved the following result.

Lemma 2.1 *Let E be a real uniformly smooth Banach space. Then, E is q-uniformly smooth if and only if there exists a constants $c_q > 0$, such that for all $x, y \in E$,*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q.$$

We recall some definitions needed later, for more details, please see Refs. [3,4,9,10] and the references therein.

Definition 2.1 Let E be a real uniformly smooth Banach space, and $T, A\colon E \longrightarrow E$ be two single-valued operators. T is said to be

(1) accretive if

$$\langle T(x) - T(y), J_q(x-y)\rangle \geq 0, \quad \forall x, y \in E;$$

(2) strictly accretive if T is accretive and

$$\langle T(x) - T(y), J_q(x-y)\rangle = 0 \quad \text{if and only if } x = y;$$

(3) strongly accretive if there exists a constant $r > 0$ such that

$$\langle T(x) - T(y), J_q(x-y)\rangle \geq r\|x-y\|^q, \quad \forall x, y \in E;$$

(4) strongly accretive with respect to A if there exists a constant $r > 0$ such that

$$\langle T(x) - T(y), J_q(A(x) - A(y))\rangle \geq r\|x-y\|^q, \quad \forall x, y \in E;$$

(5) Lipschitz continuous if there exists a constant $s > 0$ such that

$$\|T(x) - T(y)\| \leq s\|x-y\|, \quad \forall x, y \in E.$$

Definition 2.2 Let E be a real uniformly smooth Banach space, T: $E \longrightarrow E$ and η: $E \times E \longrightarrow E$ be two single-valued operators. T is said to be

(1) η-accretive if

$$\langle T(x) - T(y), J_q(\eta(x,y))\rangle \geq 0, \quad \forall x, y \in E;$$

(2) strictly η-accretive if T is η-accretive and

$$\langle T(x) - T(y), J_q(\eta(x,y))\rangle = 0 \quad \text{if and only if } x = y;$$

(3) strongly η-accretive if there exists a constant $r > 0$ such that

$$\langle T(x) - T(y), J_q(\eta(x,y))\rangle \geq r\|x-y\|^q, \quad \forall x, y \in E;$$

(4) relaxed η-accretive if there exists a constant $\alpha > 0$ such that

$$\langle T(x) - T(y), J_q(\eta(x,y))\rangle \geq -\alpha\|x-y\|^q, \quad \forall x, y \in E.$$

Definition 2.3 Let η: $E \times E \longrightarrow E$, H: $E \longrightarrow E$ be single-valued operators and M: $E \longrightarrow 2^E$ be a multi-valued operator. M is said to be

(1) accretive if

$$\langle u - v, J_q(x-y)\rangle \geq 0, \forall x, y \in E, u \in M(x), v \in M(y);$$

(2) η-accretive if

$$\langle u - v, J_q(\eta(x,y))\rangle \geq 0, \forall x, y \in E, u \in M(x), v \in M(y);$$

(3) strictly η-accretive if M is η-accretive and equality holds if and only if $x = y$;
(4) strongly η-accretive if there exists a constant $r > 0$ such that if

$$\langle u - v, J_q(\eta(x,y))\rangle \geq r\|x-y\|^q, \forall x, y \in E, u \in M(x), v \in M(y);$$

(5) relaxed η-accretive if there exists a constant $\alpha > 0$ such that if

$$\langle u - v, J_q(\eta(x,y))\rangle \geq -\alpha\|x-y\|^q, \forall x, y \in E, u \in M(x), v \in M(y);$$

(6) m-accretive if M is accretive and $(I + \rho M)(E) = E$ holds for all $\rho > 0$, where I is the identity map on E;
(7) generalized η-accretive if M is η-accretive and $(I + \rho M)(E) = E$ holds for all $\rho > 0$;
(8) H-accretive if M is accretive and $(H + \rho M)(E) = E$ holds for all $\rho > 0$;
(9) (H, η)-accretive if M is η-accretive and $(H + \rho M)(E) = E$ holds for all $\rho > 0$.

Definition 2.4 Let η: $E \times E \longrightarrow E$, A: $E \longrightarrow E$ be single-valued operators and M: $E \longrightarrow 2^E$ be a multi-valued operator. M is said to be (A, η, m)-accretive if M is relaxed η-accretive with a constant m and $(A + \rho M)(E) = E$ holds for all $\rho > 0$.

Remark 2.1

(1) (A, η, m)-accretive operators is also called (A, η)-accretive operators by Lan et al. [3].

(2) The definition of $(A, \eta, 0)$-accretive operators is that of (A, η)-accretive operators in Ref. [10] with $A = P$. If $\eta(x, y) = x - y, \forall x, y \in E$, then the definition of $(A, \eta, 0)$-accretive operators becomes that of A-accretive operators in Ref. [9] with $A = H$. If $E = \mathcal{H}$ is a Hilbert space, the definition of (A, η, m)-accretive operator becomes that of (A, η, m)-monotone operators (i.e. (A, η)-monotone operators in Ref. [7]), the definition of H-accretive operators in Ref. [9] becomes that of H-monotone operators in Refs. [1,34], the definition of the (P, η)-accretive operators in Ref. [10] becomes that of (P, η)-monotone operators in Ref. [36], if $\eta(x, y) = x - y, \forall x, y \in \mathcal{H}$, then the definition of (A, η, m)-monotone operators becomes that of A-monotone operators in Ref. [35].

Definition 2.5 [5] Let η: $E \times E \longrightarrow E$ be a single-valued operator, then $\eta(.,.)$ is said to be Lipschitz continuous, if there exists a constant $\tau > 0$ such that

$$\|\eta(u, v)\| \leq \tau \|u - v\|, \quad \forall u, v \in E.$$

Definition 2.6 [3] Let η: $E \times E \longrightarrow E$ be a single-valued operator, A: $E \longrightarrow E$ be a strictly η-accretive single-valued operator, and M: $E \longrightarrow 2^E$ be an (A, η, m)-accretive operator, $m > 0$ and $\lambda > 0$ be constants. The resolvent operator $R^{A,\eta}_{M,\lambda,m}$: $E \longrightarrow E$ associated with A, η, m, M, λ is defined by

$$R^{A,\eta}_{M,\lambda,m}(u) = (A + \lambda M)^{-1}(u), \quad \forall u \in E.$$

We also need the following result obtained by Lan et al. [3].

Lemma 2.2 *Let η: $E \times E \longrightarrow E$ be a Lipschitz continuous operator with a constant τ, A: $E \longrightarrow E$ be a strongly η-accretive operator with a constant γ and M: $E \longrightarrow 2^E$ be an (A, η, m)-accretive operator. Then, the resolvent operator $R^{A,\eta}_{M,\lambda,m}$: $E \longrightarrow E$ is Lipschitz continuous with a constant $\frac{\tau^{q-1}}{\gamma - m\lambda}$, i.e.*

$$\|R^{A,\eta}_{M,\lambda,m}(x) - R^{A,\eta}_{M,\lambda,m}(y)\| \leq \frac{\tau^{q-1}}{\gamma - m\lambda} \|x - y\|, \quad \forall x, y \in E.$$

We extend some definitions in Refs. [6,45] to more general cases as follows.

Definition 2.7 Let $E_1, E_2, \ldots, E_p$ be Banach spaces, g_1: $E_1 \longrightarrow E_1$ and N_1: $\prod_{j=1}^p E_j \longrightarrow E_1$ be two single-valued mappings.

(1) N_1 is said to be Lipschitz continuous in the first argument if there exists a constant $\xi > 0$ such that

$$\|N_1(x_1, x_2, \ldots, x_p) - N_1(y_1, x_2, \ldots, x_p)\| \leq \xi \|x_1 - y_1\|, \quad \forall x_1, y_1 \in E_1,$$
$$x_j \in E_j (j = 2, 3, \ldots, p).$$

(2) N_1 is said to be accretive in the first argument if

$$\langle N_1(x_1, x_2, \ldots, x_p) - N_1(y_1, x_2, \ldots, x_p), J_q(x_1 - y_1)\rangle \geq 0, \quad \forall x_1, y_1 \in E_1,$$
$$x_j \in E_j (j = 2, 3, \ldots, p).$$

(3) N_1 is said to be strongly accretive in the first argument if there exists a constant $\alpha > 0$ such that

$$\langle N_1(x_1, x_2, \ldots, x_p) - N_1(y_1, x_2, \ldots, x_p), J_q(x_1 - y_1)\rangle \geq \alpha \|x_1 - y_1\|^q, \quad \forall x_1, y_1 \in E_1,$$
$$x_j \in E_j (j = 2, 3, \ldots, p).$$

(4) N_1 is said to be accretive with respect to g in the first argument if

$$\langle N_1(x_1,x_2,\ldots,x_p) - N_1(y_1,x_2,\ldots,x_p), J_q(g(x_1) - g(y_1))\rangle \geq 0, \quad \forall x_1,y_1 \in E_1, \\ x_j \in E_j (j = 2,3,\ldots,p).$$

(5) N_1 is said to be strongly accretive with respect to g in the first argument if there exists a constant $\beta > 0$ such that

$$\langle N_1(x_1,x_2,\ldots,x_p) - N_1(y_1,x_2,\ldots,x_p), J_q(g(x_1) - g(y_1)) \geq \beta\|x_1 - y_1\|^q, \\ \forall x_1,y_1 \in E_1, \quad x_j \in E_j (j = 2,3,\ldots,p).$$

In a similar way, we can define the Lipschitz continuity and the strong accretivity (accretivity) of N_i: $\prod_{j=1}^{p} E_j \longrightarrow E_i$ (with respect to g_i: $E_i \longrightarrow E_i$) in the ith argument $(i = 2,3,\ldots,p)$.

3 A system of variational inclusions and a p-step iterative algorithm

In this section, we will introduce a new system of variational inclusions with (A,η,m)-accretive operators. In what follows, unless other specified, for each $i = 1,2,\ldots,p$, we always suppose that E_i is a real q-uniformly smooth Banach space, A_i, g_i: $E_i \longrightarrow E_i$, η_i: $E_i \times E_i \longrightarrow E_i$, F_i, G_i: $\prod_{j=1}^{p} E_j \longrightarrow E_i$ are single-valued mappings, M_i: $E_i \longrightarrow 2^{E_i}$ is an (A_i,η_i,m_i)-accretive operator. We consider the following problem of finding $(x_1,x_2,\ldots,x_p) \in \prod_{i=1}^{p} E_i$ such that for each $i = 1,2,\ldots,p$,

$$0 \in F_i(x_1,x_2,\ldots,x_p) + G_i(x_1,x_2,\ldots,x_p) + M_i(g_i(x_i)). \tag{3.1}$$

The problem (3.1) is called a system of variational inclusions with (A,η,m)-accretive operators. Below are some special cases of problem (3.1).

(1) For each $j = 1,2,\ldots,p$, if $E_j = \mathcal{H}_j$ is a Hilbert space, then problem (3.1) becomes the following system of variational inclusions with (A,η,m)-monotone operators, which is to find $(x_1,x_2,\ldots,x_p) \in \prod_{i=1}^{p} E_i$ such that for each $i = 1,2,\ldots,p$,

$$0 \in F_i(x_1,x_2,\ldots,x_p) + G_i(x_1,x_2,\ldots,x_p) + M_i(g_i(x_i)). \tag{3.2}$$

(2) For each $j = 1,2,\ldots,p$, if $g_j \equiv I_j$ (the identity map on E_j) and $G_j \equiv 0$, then problem (3.1) reduces to the system of variational inclusions with (A,η,m)-accretive operators, which is to find $(x_1,x_2,\ldots,x_p) \in \prod_{j=1}^{p} E_j$ such that for each $i = 1,2,\ldots,p$,

$$0 \in F_i(x_1,x_2,\ldots,x_p) + M_i(x_i). \tag{3.3}$$

(3) If $p = 1$, then problem (3.2) becomes the following variational inclusion with an (A_1,η_1,m_1)-monotone operator, which is to find $x_1 \in \mathcal{H}_1$ such that

$$0 \in F_1(x_1) + G_1(x_1) + M_1(g_1(x_1)). \tag{3.4}$$

Moreover, if $\eta_1(x_1,y_1) = x_1 - y_1$ for all $x_1, y_1 \in \mathcal{H}_1$ and $A_1 = I_1$ (the identity map on $\mathcal{H}_1$) and $m_1 = 0$, then problem (3.4) becomes the variational inclusion introduced and researched by Adly [11] which contains the variational inequality in Ref. [2] as a special case.

If $p = 1$, then problem (3.3) becomes the following variational inclusion with an (A_1, η_1, m_1)-accretive operator, which is to find $x_1 \in E_1$ such that

$$0 \in F_1(x_1) + M_1(x_1). \tag{3.5}$$

Problem (3.5) contains the variational inclusions in Refs. [1,9] as special cases.
If $p = 2$, then Problem (3.3) becomes the following system of variational inclusions with (A, η, m)-monotone operators, which is to find $(x_1, x_2) \in E_1 \times E_2$ such that

$$\begin{aligned} 0 &\in F_1(x_1, x_2) + M_1(x_1), \\ 0 &\in F_2(x_1, x_2) + M_2(x_2). \end{aligned} \tag{3.6}$$

Problem (3,6) contains the system of variational inclusions with H-monotone operators in Ref. [34], the system of variational inclusions with A-monotone operators in Ref. [35], the system of variational inclusions with (H, η)-monotone operators in Ref. [36] as special cases.

(4) For each $j = 1, 2, \ldots, p$, if $E_j = \mathcal{H}_j$ is a Hilbert space, and $M_j(x_j) = \Delta_{\eta_j}\varphi_j$ for all $x_j \in \mathcal{H}_j$, where φ_j: $\mathcal{H}_j \longrightarrow R \cup \{+\infty\}$ is a proper, η_j-subdifferentiable functional and $\Delta_{\eta_j}\varphi_j$ denotes the η_j-subdifferential operator of φ_j, then problem (3.3) reduces to the following system of variational-like inequalities, which is to find $(x_1, x_2, \ldots, x_p) \in \prod_{i=1}^{p} \mathcal{H}_i$ such that for each $i = 1, 2, \ldots, p$,

$$\langle F_i(x_1, x_2, \ldots, x_p), \eta_i(z_i, x_i)\rangle + \varphi_i(z_i) - \varphi_i(x_i) \geq 0, \quad \forall z_i \in \mathcal{H}_i. \tag{3.7}$$

(5) For each $j = 1, 2, \ldots, p$, if $E_j = \mathcal{H}_j$ is a Hilbert space, and $M_j(x_j) = \partial\varphi_j(x_j)$, for all $x_j \in \mathcal{H}_j$, where φ_j: $\mathcal{H}_j \longrightarrow R \cup \{+\infty\}$ is a proper, convex, lower semicontinuous functional and $\partial\varphi_j$ denotes the subdifferential operator of φ_j, then problem (3.3) reduces to the following system of variational inequalities, which is to find $(x_1, x_2, \ldots, x_p) \in \prod_{i=1}^{p} \mathcal{H}_i$ such that for each $i = 1, 2, \ldots, p$,

$$\langle F_i(x_1, x_2, \ldots, x_p), z_i - x_i\rangle + \varphi_i(z_i) - \varphi_i(x_i) \geq 0, \forall z_i \in \mathcal{H}_i. \tag{3.8}$$

(6) For each $j = 1, 2, \ldots, p$, if $M_j(x_j) = \partial\delta_{K_j}(x_j)$ for all $x_j \in \mathcal{H}_j$, where $K_j \subset \mathcal{H}_j$ is a nonempty, closed and convex subsets and δ_{K_j} denotes the indicator of K_j, then problem (3.8) reduces to the following system of variational inequalities, which is to find $(x_1, x_2, \ldots, x_p) \in \prod_{i=1}^{p} \mathcal{H}_i$ such that for each $i = 1, 2, \ldots, p$,

$$\langle F_i(x_1, x_2, \ldots, x_p), z_i - x_i\rangle \geq 0, \quad \forall z_i \in K_i. \tag{3.9}$$

Problem (3.9) was introduced and researched in Refs. [15,27–29]. If $p = 2$, then problem (3.7), (3.8) and (3.9), respectively, become the problem (3.2), (3.3) and (3.4) in Ref.[36]. It is easy to see that problem (3.4) in Ref. [36] contains the models of system of variational inequalities in Refs. [20–24] as special cases.

It is worthy noting that problem (3.1)–(3.8) are all new problems.

4 Existence and uniqueness of the solution

In this section, we will prove existence and uniqueness for solutions of problem (3.1). For our main results, we give a characterization of the solution of problem (3.1) as follows.

Lemma 4.1 *For $i = 1, 2, \ldots, p$, let η_i: $E_i \times E_i \longrightarrow E_i$ be a single-valued operator, A_i: $E_i \longrightarrow E_i$ be a strictly η_i-accretive operator and M_i: $E_i \longrightarrow 2^{E_i}$ be an (A_i, η_i, m_i)-accretive operator. Then $(x_1, x_2, \ldots, x_p) \in \prod_{i=1}^{p} E_i$ is a solution of the problem (3.1) if and only if for each $i = 1, 2, \ldots, p$,*

$$g_i(x_i) = R^{A_i,\eta_i}_{M_i,\lambda_i,m_i}(A_i(g_i(x_i)) - \lambda_i F_i(x_1, x_2, \ldots, x_p) - \lambda_i G_i(x_1, x_2, \ldots, x_p)),$$

where $R^{A_i,\eta_i}_{M_i,\lambda_i,m_i} = (A_i + \lambda_i M_i)^{-1}$, $m_i > 0$ and $\lambda_i > 0$ are constants.

Proof The fact directly follows from Definition 2.6. □

Let $\Gamma = \{1, 2, \ldots, p\}$.

Theorem 4.1 *For $i = 1, 2, \ldots, p$, let η_i: $E_i \times E_i \to E_i$ be Lipschitz continuous with a constant σ_i, A_i: $E_i \to E_i$ be strongly η_i-accretive and Lipschitz continuous with constants γ_i and τ_i, respectively, g_i: $E_i \to E_i$ be strongly accretive and Lipschitz continuous with constants β_i and θ_i, respectively, M_i: $E_i \to 2^{E_i}$ be an (A_i, η_i, m_i)-accretive operator, let F_i: $\prod_{j=1}^{p} E_j \to E_i$ be a single-valued mapping such that F_i is strongly accretive with respect to $\hat{g}_i$ in the ith argument with a constant r_i and Lipschitz continuous in the ith argument with a constant s_i, where $\hat{g}_i$: $E_i \to E_i$ is defined by $\hat{g}_i(x_i) = A_i \circ g_i(x_i) = A_i(g_i(x_i)), \forall x_i \in E_i$, F_i is Lipschitz continuous in the jth argument with a constant t_{ij} for each $j \in \Gamma, j \neq i$, G_i: $\prod_{j=1}^{p} E_j \to E_i$ be a single-valued mapping such that G_i is Lipschitz continuous in the jth argument with a constant l_{ij} for each $j \in \Gamma$. If there exist constants $\lambda_i > 0 (i = 1, 2, \ldots, p)$ such that,*

$$\begin{aligned}
&\sqrt[q]{1 - q\beta_1 + c_q\theta_1^q} + \frac{\sigma_1^{q-1}}{\gamma_1 - \lambda_1 m_1}\sqrt[q]{\tau_1^q\theta_1^q - q\lambda_1 r_1 + c_q\lambda_1{}^q s_1^q} + \frac{l_{11}\lambda_1\sigma_1^{q-1}}{\gamma_1 - \lambda_1 m_1} \\
&\qquad + \sum_{k=2}^{p} \frac{\lambda_k\sigma_k^{q-1}}{\gamma_k - \lambda_k m_k}(t_{k1} + l_{k1}) < 1, \\
&\sqrt[q]{1 - q\beta_2 + c_q\theta_2^q} + \frac{\sigma_2^{q-1}}{\gamma_2 - \lambda_2 m_2}\sqrt[q]{\tau_2^q\theta_2^q - q\lambda_2 r_2 + c_q\lambda_2{}^q s_2^q} + \frac{l_{22}\lambda_2\sigma_2^{q-1}}{\gamma_2 - \lambda_2 m_2} \\
&\qquad + \sum_{k\in\Gamma, k\neq 2} \frac{\lambda_k\sigma_k^{q-1}}{\gamma_k - \lambda_k m_k}(t_{k2} + l_{k2}) < 1, \\
&\qquad \ldots, \\
&\sqrt[q]{1 - q\beta_p + c_q\theta_p^q} + \frac{\sigma_p^{q-1}}{\gamma_p - \lambda_p m_p}\sqrt[q]{\tau_p^q\theta_p^q - q\lambda_p r_p + c_q\lambda_p{}^q s_p^q} + \frac{l_{pp}\lambda_p\sigma_p^{q-1}}{\gamma_p - \lambda_p m_p} \\
&\qquad + \sum_{k=1}^{p-1} \frac{\sigma_k^{q-1}\lambda_k}{\gamma_k - \lambda_k m_k}(t_{k,p} + l_{k,p}) < 1.
\end{aligned} \tag{4.1}$$

Then, problem (3.1) admits a unique solution.

Proof For $i = 1, 2, \ldots, p$ and for any given $\lambda_i > 0$, define a single-valued mapping T_{i,λ_i}: $\prod_{j=1}^{p} E_j \to E_i$ by □

$$
\begin{aligned}
T_{i,\lambda_i}(x_1,x_2,\ldots,x_p) = \ & x_i - g_i(x_i) + R^{A_i,\eta_i}_{M_i,\lambda_i,m_i}(A_i(g_i(x_i)) \\
& -\lambda_i F_i(x_1,x_2,\ldots,x_p) - \lambda_i G_i(x_1,x_2,\ldots,x_p)),
\end{aligned} \tag{4.2}
$$

for any $(x_1,x_2,\ldots,x_p) \in \prod_{i=1}^p E_i$.

For any $(x_1,x_2,\ldots,x_p),(y_1,y_2,\ldots,y_p) \in \prod_{i=1}^p E_i$, it follows from (4.2) that for $i=1,2,\ldots,p$,

$$
\begin{aligned}
&\|T_{i,\lambda_i}(x_1,x_2,\ldots,x_p) - T_{i,\lambda_i}(y_1,y_2,\ldots,y_p)\|_i \\
&\quad = \|x_i - g_i(x_i) + R^{A_i,\eta_i}_{M_i,\lambda_i,m_i}(A_i(g_i(x_i)) - \lambda_i F_i(x_1,x_2,\ldots,x_p) - \lambda_i G_i(x_1,x_2,\ldots,x_p)) \\
&\qquad -[y_i - g_i(y_i) + R^{A_i,\eta_i}_{M_i,\lambda_i,m_i}(A_i(g_i(y_i)) - \lambda_i F_i(y_1,y_2,\ldots,y_p) - \lambda_i G_i(y_1,y_2,\ldots,y_p))]\|_i \\
&\quad \le \|x_i - y_i - (g_i(x_i) - g_i(y_i))\|_i + \|R^{A_i,\eta_i}_{M_i,\lambda_i,m_i}(A_i(g_i(x_i)) - \lambda_i F_i(x_1,x_2,\ldots,x_p) \\
&\qquad -\lambda_i G_i(x_1,x_2,\ldots,x_p)) - R^{A_i,\eta_i}_{M_i,\lambda_i,m_i}(A_i(g_i(y_i)) - \lambda_i F_i(y_1,y_2,\ldots,y_p) \\
&\qquad -\lambda_i G_i(y_1,y_2,\ldots,y_p))\|_i.
\end{aligned} \tag{4.3}
$$

For $i=1,2,\ldots,p$, since g_i is strongly accretive and Lipschitz continuous with constants β_i and θ_i, respectively, we have

$$
\begin{aligned}
& \|x_i - y_i - (g_i(x_i) - g_i(y_i))\|_i^q \\
= \ & \|x_i - y_i\|_i^q - q\langle g_i(x_i) - g_i(y_i), J_q(x_i - y_i)\rangle + c_q\|g_i(x_i) - g_i(y_i)\|_i^q \\
\le \ & (1 - q\beta_i + c_q\theta_i^q)\|x_i - y_i\|_i^q,
\end{aligned} \tag{4.4}
$$

It follows from Lemma 2.1 that for $i=1,2,\ldots,p$,

$$
\begin{aligned}
&\|R^{A_i,\eta_i}_{M_i,\lambda_i,m_i}(A_i(g_i(x_i)) - \lambda_i F_i(x_1,x_2,\ldots,x_p) - \lambda_i G_i(x_1,x_2,\ldots,x_p)) \\
&\quad -R^{A_i,\eta_i}_{M_i,\lambda_i,m_i}(A_i(g_i(y_i)) - \lambda_i F_i(y_1,y_2,\ldots,y_p) - \lambda_i G_i(y_1,y_2,\ldots,y_p))\|_i \\
&\quad \le \frac{\sigma_i^{q-1}}{\gamma_i - \lambda_i m_i}\|(A_i(g_i(x_i)) - A_i(g_i(y_i))) - \lambda_i(F_i(x_1,x_2,\ldots,x_p) - F_i(y_1,y_2,\ldots,y_p))\|_i \\
&\quad + \frac{\sigma_i^{q-1}\lambda_i}{\gamma_i - \lambda_i m_i}\|G_i(x_1,x_2,\ldots,x_p) - G_i(y_1,y_2,\ldots,y_p)\|_i \le \frac{\sigma_i^{q-1}}{\gamma_i - \lambda_i m_i}\|A_i(g_i(x_i)) \\
&\quad -A_i(g_i(y_i)) - \lambda_i(F_i(x_1,x_2,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_p) - F_i(x_1,x_2,\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_p))\|_i \\
&\quad + \frac{\sigma_i^{q-1}\lambda_i}{\gamma_i - \lambda_i m_i}\left(\sum_{j\in\Gamma, j\ne i}\|F_i(y_1,y_2,\ldots,y_{j-1},x_j,x_{j+1},\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_p)\right. \\
&\quad \left. -F_i(y_1,y_2,\ldots,y_{j-1},y_j,x_{j+1},\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_p)\|_i\right) + \frac{\sigma_i^{q-1}\lambda_i}{\gamma_i - \lambda_i m_i} \\
&\quad \left(\sum_{j=1}^p \|G_i(y_1,y_2,\ldots,y_{j-1},x_j,x_{j+1},\ldots,x_p) - G_i(y_1,y_2,\ldots,y_{j-1},y_j,x_{j+1},\ldots,x_p)\|_i\right).
\end{aligned} \tag{4.5}
$$

For $i=1,2,\ldots,p$, since A_i is Lipschitz continuous with a constant τ_i, and g_i is Lipschitz continuous with a constant θ_i and F_i is $\hat{g}_i$-strongly accretive in the ith argument with a constant r_i and Lipschitz continuous in the ith argument with a constant s_i, we have

$$
\begin{aligned}
&\|A_i(g_i(x_i)) - A_i(g_i(y_i)) - \lambda_i(F_i(x_1,x_2,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_p)\\
&\quad -F_i(x_1,x_2,\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_p))\|_i^q \le \|(A_i(g_i(x_i)) - A_i(g_i(y_i)))\|_i^q\\
&\quad -q\lambda_i\langle F_i(x_1,x_2,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_p) - F_i(x_1,x_2,\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_p),\\
&\quad A_i(g_i(x_i)) - A_i(g_i(y_i))\rangle + c_q{\lambda_i}^q\|F_i(x_1,x_2,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_p)\\
&\quad -F_i(x_1,x_2,\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_p)\|_i^q \le \tau_i^q\|g_i(x_i) - g_i(y_i)\|_i^q - q\lambda_i r_i\|x_i - y_i\|_i^q\\
&\quad +c_q{\lambda_i}^q s_i^q\|x_i - y_i\|_i^q \le (\tau_i^q\theta_i^q - q\lambda_i r_i + c_q{\lambda_i}^q s_i^q)\|x_i - y_i\|_i^q.
\end{aligned}
\tag{4.6}
$$

For $i = 1,2,\ldots,p$, since F_i is Lipschitz continuous in the jth arguments with a constant t_{ij} $(j \in \Gamma, j \neq i)$, we have

$$
\begin{aligned}
&\|F_i(y_1,y_2,\ldots,y_{j-1},x_j,x_{j+1},\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_p) - F_i(y_1,y_2,\ldots,y_{j-1},y_j,x_{j+1},\ldots,\\
&\quad x_{i-1},y_i,x_{i+1},\ldots,x_p)\|_i \le t_{ij}\|x_j - y_j\|_j.
\end{aligned}
\tag{4.7}
$$

For $i = 1,2,\ldots,p$, since G_i is Lipschitz continuous in the jth arguments with a constant l_{ij} $(j = 1,2,\ldots,p)$, we have

$$
\begin{aligned}
&\|G_i(y_1,y_2,\ldots,y_{j-1},x_j,x_{j+1},\ldots,x_p) - G_i(y_1,y_2,\ldots,y_{j-1},y_j,x_{j+1},\ldots,x_p)\\
&\quad \|_i \le l_{ij}\|x_j - y_j\|_j.
\end{aligned}
\tag{4.8}
$$

It follows from (4.3) to (4.8) that for each $i = 1,2,\ldots,p$

$$
\begin{aligned}
&\|T_{i,\lambda_i}(x_1,x_2,\ldots,x_p) - T_{i,\lambda_i}(y_1,y_2,\ldots,y_p)\|_i \le (\sqrt[q]{1 - q\beta_i + c_q\theta_i^q} + \frac{\sigma_i^{q-1}}{\gamma_i - \lambda_i m_i}\\
&\quad \sqrt[q]{{\tau_i}^q\theta_i^q - q\lambda_i r_i + c_q{\lambda_i}^q s_i^q} + \frac{l_{ii}\lambda_i\sigma_i^{q-1}}{\gamma_i - \lambda_i m_i})\|x_i - y_i\|_i\\
&\quad +\frac{\lambda_i\sigma_i^{q-1}}{\gamma_i - \lambda_i m_i}\left[\sum_{j\in\Gamma, j\neq i}(t_{ij} + l_{ij})\|x_j - y_j\|_j\right].
\end{aligned}
\tag{4.9}
$$

Hence,

$$
\begin{aligned}
&\sum_{i=1}^{p} \|T_{i,\lambda_i}(x_1,x_2,\ldots,x_p) - T_{i,\lambda_i}(y_1,y_2,\ldots,y_p)\|_i \le \sum_{i=1}^{p}\left\{\left(\sqrt[q]{1 - q\beta_i + c_q\theta_i^q} + \frac{\sigma_i^{q-1}}{\gamma_i - \lambda_i m_i}\right.\right.\\
&\quad \left.\sqrt[q]{{\tau_i}^q\theta_i^q - q\lambda_i r_i + c_q{\lambda_i}^q s_i^q} + \frac{l_{ii}\lambda_i\sigma_i^{q-1}}{\gamma_i - \lambda_i m_i}\right)\|x_i - y_i\|_i + \frac{\lambda_i\sigma_i^{q-1}}{\gamma_i - \lambda_i m_i}
\end{aligned}
$$

$$
\left[\sum_{j\in\Gamma,j\neq i}(t_{ij}+l_{ij})\|x_j-y_j\|_j\right]\Bigg\} = \Bigg(\sqrt[q]{1-q\beta_1+c_q\theta_1^q}+\frac{\sigma_1^{q-1}}{\gamma_1-\lambda_1 m_1}
$$

$$
\sqrt[q]{\tau_1^q\theta_1^q-q\lambda_1 r_1+c_q\lambda_1{}^q s_1^q}+\frac{l_{11}\lambda_1\sigma_1^{q-1}}{\gamma_1-\lambda_1 m_1}+\sum_{k=2}^{p}\frac{\lambda_k\sigma_k^{q-1}}{\gamma_k-\lambda_k m_k}(t_{k1}+l_{k1})\Bigg)\|x_1-y_1\|_1
$$

$$
+\Bigg(\sqrt[q]{1-q\beta_2+c_q\theta_2^q}+\frac{\sigma_2^{q-1}}{\gamma_2-\lambda_2 m_2}\sqrt[q]{\tau_2^q\theta_2^q-q\lambda_2 r_2+c_q\lambda_2{}^q s_2^q}+\frac{l_{22}\lambda_2\sigma_2^{q-1}}{\gamma_2-\lambda_2 m_2}
$$

$$
+\sum_{k\in\Gamma,k\neq 2}\frac{\lambda_k\sigma_k^{q-1}}{\gamma_k-\lambda_k m_k}(t_{k2}+l_{k2})\Bigg)\|x_2-y_2\|_2
$$

$$
+\dots
$$

$$
+\Bigg(\sqrt[q]{1-q\beta_p+c_q\theta_p^q}+\frac{\sigma_p^{q-1}}{\gamma_p-\lambda_p m_p}\sqrt[q]{\tau_p^q\theta_p^q-q\lambda_p r_p+c_q\lambda_p{}^q s_p^q}+\frac{l_{pp}\lambda_p\sigma_p^{q-1}}{\gamma_p-\lambda_p m_p}
$$

$$
+\sum_{k=1}^{p-1}\frac{\sigma_k^{q-1}\lambda_k}{\gamma_k-\lambda_k m_k}(t_{k,p}+l_{k,p})\Bigg)\|x_p-y_p\|_p\leq\xi\left(\sum_{k=1}^{p}\|x_k-y_k\|_k\right), \tag{4.10}
$$

where

$$
\xi=\max\Bigg\{\sqrt[q]{1-q\beta_1+c_q\theta_1^q}+\frac{\sigma_1^{q-1}}{\gamma_1-\lambda_1 m_1}\sqrt[q]{\tau_1^q\theta_1^q-q\lambda_1 r_1+c_q\lambda_1{}^q s_1^q}
$$

$$
+\frac{l_{11}\lambda_1\sigma_1^{q-1}}{\gamma_1-\lambda_1 m_1}+\sum_{k=2}^{p}\frac{\lambda_k\sigma_k^{q-1}}{\gamma_k-\lambda_k m_k}(t_{k1}+l_{k1}),\ \sqrt[q]{1-q\beta_2+c_q\theta_2^q}+\frac{\sigma_2^{q-1}}{\gamma_2-\lambda_2 m_2}
$$

$$
\sqrt[q]{\tau_2^q\theta_2^q-q\lambda_2 r_2+c_q\lambda_2{}^q s_2^q}+\frac{l_{22}\lambda_2\sigma_2^{q-1}}{\gamma_2-\lambda_2 m_2}+\sum_{k\in\Gamma,k\neq 2}\frac{\lambda_k\sigma_k^{q-1}}{\gamma_k-\lambda_k m_k}(t_{k2}+l_{k2}),
$$

$$
\dots,
$$

$$
\sqrt[q]{1-q\beta_p+c_q\theta_p^q}+\frac{\sigma_p^{q-1}}{\gamma_p-\lambda_p m_p}\sqrt[q]{\tau_p^q\theta_p^q-q\lambda_p r_p+c_q\lambda_p{}^q s_p^q}+\frac{l_{pp}\lambda_p\sigma_p^{q-1}}{\gamma_p-\lambda_p m_p}
$$

$$
+\sum_{k=1}^{p-1}\frac{\sigma_k^{q-1}\lambda_k}{\gamma_k-\lambda_k m_k}(t_{k,p}+l_{k,p})\Bigg\}.
$$

Define $\|\cdot\|_\Gamma$ on $\prod_{i=1}^p E_i$ by $\|(x_1,x_2,\dots,x_p)\|_\Gamma=\|x_1\|_1+\|x_2\|_2+\dots+\|x_p\|_p$, $\forall(x_1,x_2,\dots,x_p)\in\prod_{i=1}^p E_i$. It is easy to see that $\prod_{i=1}^p E_i$ is a Banach space. For any given $\lambda_i>0 (i\in\Gamma)$, define $W_{\Gamma,\lambda_1,\lambda_2,\dots,\lambda_p}\colon \prod_{i=1}^p E_i\to\prod_{i=1}^p E_i$ by

$$
W_{\Gamma,\lambda_1,\lambda_2,\dots,\lambda_p}(x_1,x_2,\dots,x_p)=(T_{1,\lambda_1}(x_1,x_2,\dots,x_p),T_{2,\lambda_2}(x_1,x_2,\dots,x_p)),\dots,
$$

$$
T_{p,\lambda_p}(x_1,x_2,\dots,x_p)),\ \text{for all }(x_1,x_2,\dots,x_p)\in\prod_{i=1}^p E_i.
$$

By (4.1), we know that $0 < \xi < 1$, it follows from (4.10) that

$$\|W_{\Gamma,\lambda_1,\lambda_2,\ldots,\lambda_p}(x_1,x_2,\ldots,x_p) - W_{\Gamma,\lambda_1,\lambda_2,\ldots,\lambda_p}(x_1,x_2,\ldots,x_p)\|_\Gamma \le \xi\|(x_1,x_2,\ldots,x_p) - (y_1,y_2,\ldots,y_p)\|_\Gamma.$$

This shows that $W_{\Gamma,\lambda_1,\lambda_2,\ldots,\lambda_p}$ is a contraction operator. Hence, there exists a unique $(x_1,x_2,\ldots,x_p) \in \prod_{i=1}^p E_i$, such that

$$W_{\Gamma,\lambda_1,\lambda_2,\ldots,\lambda_p}(x_1,x_2,\ldots,x_p) = (x_1,x_2,\ldots,x_p),$$

that is, for $i = 1,2,\ldots,p$,

$$g_i(x_i) = R^{A_i,\eta_i}_{M_i,\lambda_i,m_i}(A_i(g_i(x_i)) - \lambda_i F_i(x_1,x_2,\ldots,x_p) - \lambda_i G_i(x_1,x_2,\ldots,x_p)).$$

By lemma 4.1, $(x_1,x_2,\ldots,x_p)$ is the unique solution of problem (3.1).This completes this proof.

5 Iterative algorithm and convergence

In this section, we will construct a new multi-step iterative algorithm for approximating the unique solution of problem (3.1) and discuss the convergence analysis of this Algorithm.

Lemma 5.1 [36] *Let $\{c_n\}$ and $\{k_n\}$ be two real sequences of non-negative numbers that satisfy the following conditions.*

(1) $0 \le k_n < 1, n = 0,1,2,\ldots$ *and* $\limsup_n k_n < 1$,
(2) $c_{n+1} \le k_n c_n, n = 0,1,2,\ldots,$ *then c_n converges to 0 as $n \to \infty$.*

Algorithm 5.1 For $i = 1,2,\ldots,p$, let $A_i, M_i, F_i, g_i, \eta_i$ be the same as in Theorem 4.1. For any given $(x_1^0, x_2^0,\ldots,x_p^0) \in \prod_{j=1}^p E_j$, define a multi-step iterative sequence $\{(x_1^n, x_2^n,\ldots,x_p^n))\}$ by

$$x_i^{n+1} = \alpha_n x_i^n + (1-\alpha_n)[x_i^n - g_i(x_i^n) + R^{A_i,\eta_i}_{M_i,\lambda_i,m_i}(A_i(g_i(x_i^n)) - \lambda_i F_i(x_1^n,x_2^n,\ldots,x_p^n) - \lambda_i G_i(x_1^n,x_2^n,\ldots,x_p^n))], \tag{5.1}$$

where

$$0 \le \alpha_n < 1 \quad \text{and} \ \limsup_n \alpha_n < 1. \tag{5.2}$$

Theorem 5.1 *For $i = 1,2,\ldots,p$, let $A_i, M_i, F_i, g_i, \eta_i$ be the same as in Theorem 4.1. Assume that all the conditions of Theorem 4.1 hold. Then $\{(x_1^n, x_2^n,\ldots,x_p^n))\}$ generated by Algorithm 5.1 converges strongly to the unique solution $(x_1,x_2,\ldots,x_p)$ of problem (3.1).*

Proof By Theorem 4.1, problem (3.1) admits a unique solution $(x_1,x_2,\ldots,x_p)$, it follows from Lemma 4.1 that for each $i = 1,2,\ldots,p$, □

$$g_i(x_i) = R^{A_i,\eta_i}_{M_i,\lambda_i,m_i}(A_i(g_i(x_i)) - \lambda_i F_i(x_1, x_2, \dots, x_p) - \lambda_i G_i(x_1, x_2, \dots, x_p)). \tag{5.3}$$

It follows from (5.1) and (5.3) that for each $i = 1, 2, \dots, p$,

$$\begin{aligned}\|x_i^{n+1} - x_i\|_i = {} & \|\alpha_n(x_i^n - x_i) + (1-\alpha_n)[x_i^n - g_i(x_i^n) - (x_i - g_i(x_i))\\ & + R^{A_i,\eta_i}_{M_i,\lambda_i,m_i}(A_i(g_i(x_i^n)) - \lambda_i F_i(x_1^n, x_2^n, \dots, x_p^n) - \lambda_i G_i(x_1^n, x_2^n, \dots, x_p^n))\\ & - R^{A_i,\eta_i}_{M_i,\lambda_i,m_i}(A_i(g_i(x_i)) - \lambda_i F_i(x_1, x_2, \dots, x_p) - \lambda_i G_i(x_1, x_2, \dots, x_p))]\|_i\\ & \le \alpha_n\|x_i^n - x_i\|_i + (1-\alpha_n)\|x_i^n - g_i(x_i^n) - (x_i - g_i(x_i))\|_i\\ & + (1-\alpha_n)\|R^{A_i,\eta_i}_{M_i,\lambda_i,m_i}(A_i(g_i(x_i^n)) - \lambda_i F_i(x_1^n, x_2^n, \dots, x_p^n) - \lambda_i G_i(x_1^n, x_2^n, \dots, x_p^n))\\ & - R^{A_i,\eta_i}_{M_i,\lambda_i,m_i}(A_i(g_i(x_i)) - \lambda_i F_i(x_1, x_2, \dots, x_p) - \lambda_i G_i(x_1, x_2, \dots, x_p))\|_i. \end{aligned} \tag{5.4}$$

For $i = 1, 2, \dots, p$, since g_i is strongly accretive and Lipschitz continuous with constants β_i and θ_i, respectively, we have

$$\|x_i^n - g_i(x_i^n) - (x_i - g_i(x_i))\|_i^q \le (1 - q\beta_i + c_q\theta_i^q)\|x_i^n - x_i\|_i^q. \tag{5.5}$$

It follows from Lemma 2.1 that for $i = 1, 2, \dots, p$

$$\begin{aligned} & \|R^{A_i,\eta_i}_{M_i,\lambda_i,m_i}(A_i(g_i(x_i^n)) - \lambda_i F_i(x_1^n, x_2^n, \dots, x_p^n) - \lambda_i G_i(x_1^n, x_2^n, \dots, x_p^n))\\ & \quad - R^{A_i,\eta_i}_{M_i,\lambda_i,m_i}(A_i(g_i(x_i)) - \lambda_i F_i(x_1, x_2, \dots, x_p) - \lambda_i G_i(x_1, x_2, \dots, x_p))\|_i\\ & \quad \le \frac{\sigma_i^{q-1}}{\gamma_i - \lambda_i m_i}\|A_i(g_i(x_i^n)) - A_i(g_i(x_i)) - \lambda_i(F_i(x_1^n, x_2^n, \dots, x_{i-1}^n, x_i^n, x_{i+1}^n, \dots, x_p^n)\\ & \quad - F_i(x_1^n, x_2^n, \dots, x_{i-1}^n, x_i, x_{i+1}^n, \dots, x_p^n))\|_i + \frac{\lambda_i\sigma_i^{q-1}}{\gamma_i - \lambda_i m_i}\Bigg(\sum_{j\in\Gamma, j\ne i}\|F_i(x_1, x_2, \dots, x_{j-1}, x_j^n,\\ & \quad x_{j+1}^n, \dots, x_{i-1}^n, x_i, x_{i+1}^n, \dots, x_p^n) - F_i(x_1, x_2, \dots, x_{j-1}, x_j, x_{j+1}^n, \dots, x_{i-1}^n, x_i, x_{i+1}^n, \dots, x_p^n)\|_i\Bigg)\\ & \quad + \frac{\lambda_i\sigma_i^{q-1}}{\gamma_i - \lambda_i m_i}\Bigg(\sum_{j=1}^{p}\|G_i(x_1, x_2, \dots, x_{j-1}, x_j^n, x_{j+1}^n, \dots, x_p^n)\\ & \quad - G_i(x_1, x_2, \dots, x_{j-1}, x_j, x_{j+1}^n, \dots, x_p^n)\|_i\Bigg). \end{aligned} \tag{5.6}$$

For $i = 1, 2, \dots, p$, since A_i is Lipschitz continuous with a constant τ_i, and g_i is Lipschitz continuous with a constant θ_i and F_i is $\hat{g}_i$-strongly accretive in the i-th argument with a constant r_i and Lipschitz continuous in the i-th argument with a constant s_i, we have

$$\begin{aligned} & \|A_i(g_i(x_i^n)) - A_i(g_i(x_i)) - \lambda_i(F_i(x_1^n, x_2^n, \dots, x_{i-1}^n, x_i^n, x_{i+1}^n, \dots, x_p^n)\\ & \quad - F_i(x_1^n, x_2^n, \dots, x_{i-1}^n, x_i, x_{i+1}^n, \dots, x_p^n))\|_i^q \le (\tau_i^q\theta_i^q - q\lambda_i r_i + c_q\lambda_i^q s_i^q)\|x_i^n - x_i\|^q. \end{aligned} \tag{5.7}$$

For $i = 1, 2, \dots, p$, since F_i is Lipschitz continuous in the j-th arguments with a constant t_{ij} $(j \in \Gamma, j \ne i)$, we have

$$\begin{aligned} & \|F_i(x_1^n, x_2^n, \dots, x_{j-1}^n, x_j^n, x_{j+1}^n, \dots, x_p^n) - F_i(x_1^n, x_2^n, \dots, x_{j-1}^n, x_j, x_{j+1}^n, \dots, x_p^n)\\ & \quad \|_i \le t_{ij}\|x_j^n - x_j\|_j. \end{aligned} \tag{5.8}$$

For $i = 1, 2, \ldots, p$, since G_i is Lipschitz continuous in the j-th arguments with a constant l_{ij} ($j = 1, 2, \ldots, p$), we have

$$\begin{aligned}&\|G_i(x_1, x_2, \ldots, x_{j-1}, x_j^n, x_{j+1}^n, \ldots, x_p^n) - G_i(x_1, x_2, \ldots, x_{j-1}, x_j, x_{j+1}^n, \ldots, x_p^n)\\ &\quad\|_i \le l_{ij}\|x_j^n - x_j\|_j.\end{aligned} \tag{5.9}$$

It follows from (5.4) to (5.9) that for $i = 1, 2, \ldots, p$

$$\begin{aligned}\|x_i^{n+1} - x_i\|_i &\le \alpha_n\|x_i^n - x_i\|_i + (1-\alpha_n)\sqrt[q]{1 - q\beta_i + c_q\theta_i^q}\,\|x_i^n - x_i\|_i\\ &+(1-\alpha_n)\frac{\sigma_i^{q-1}}{\gamma_i - \lambda_i m_i}\sqrt[q]{\tau_i^q\theta_i^q - q\lambda_i r_i + c_q\lambda_i^q s_i^q}\,\|x_i^n - x_i\|_i + (1-\alpha_n)\frac{\lambda_i\sigma_i^{q-1}}{\gamma_i - \lambda_i m_i}\\ &\times\left(\sum_{j\in\Gamma, j\ne i} t_{ij}\|x_j^n - x_j\|_j\right) + (1-\alpha_n)\frac{\lambda_i\sigma_i^{q-1}}{\gamma_i - \lambda_i m_i}\left(\sum_{j=1}^p l_{ij}\|x_j^n - x_j\|_j\right)\\ &= \alpha_n\|x_i^n - x_i\|_i + (1-\alpha_n)(\sqrt[q]{1 - q\beta_i + c_q\theta_i^q} + \frac{\sigma_i^{q-1}}{\gamma_i - \lambda_i m_i}\sqrt[q]{\tau_i^q\theta_i^q - q\lambda_i r_i + c_q\lambda_i^q s_i^q}\\ &+\frac{l_{ii}\lambda_i\sigma_i^{q-1}}{\gamma_i - \lambda_i m_i})\|x_i^n - x_i\|_i + (1-\alpha_n)\frac{\sigma_i^{q-1}}{\gamma_i - \lambda_i m_i}\left(\sum_{j\in\Gamma, j\ne i}(t_{ij} + l_{ij})\|x_j^n - x_j\|_j\right).\end{aligned} \tag{5.10}$$

It follows from (5.10) that

$$\begin{aligned}\sum_{i=1}^p \|x_i^{n+1} - x_i\|_i &\le \sum_{i=1}^p\Big[\alpha_n\|x_i^n - x_i\|_i + (1-\alpha_n)(\sqrt[q]{1 - q\beta_i + c_q\theta_i^q}\\ &+\frac{\sigma_i^{q-1}}{\gamma_i - \lambda_i m_i}\sqrt[q]{\tau_i^q\theta_i^q - q\lambda_i r_i + c_q\lambda_i^q s_i^q} + \frac{l_{ii}\lambda_i\sigma_i^{q-1}}{\gamma_i - \lambda_i m_i})\|x_i^n - x_i\|_i + (1-\alpha_n)\frac{\sigma_i^{q-1}}{\gamma_i - \lambda_i m_i}\\ &\times\left(\sum_{j\in\Gamma, j\ne i}(t_{ij} + l_{ij})\|x_j^n - x_j\|_j\right)\Big] \le \alpha_n(\sum_{i=1}^p\|x_i^n - x_i\|_i) + (1-\alpha_n)\xi\left(\sum_{i=1}^p\|x_i^n - x_i\|_i\right)\\ &= (\xi + (1-\xi)\alpha_n)(\sum_{i=1}^p\|x_i^n - x_i\|_i).\end{aligned} \tag{5.11}$$

Where ξ is defined by

$$\begin{aligned}\xi = \max\Bigg\{&\sqrt[q]{1 - q\beta_1 + c_q\theta_1^q} + \frac{\sigma_1^{q-1}}{\gamma_1 - \lambda_1 m_1}\sqrt[q]{\tau_1^q\theta_1^q - q\lambda_1 r_1 + c_q{\lambda_1}^q s_1^q} + \frac{l_{11}\lambda_1\sigma_1^{q-1}}{\gamma_1 - \lambda_1 m_1}\\ &+\sum_{k=2}^p\frac{\lambda_k\sigma_k^{q-1}}{\gamma_k - \lambda_k m_k}(t_{k1} + l_{k1}), \sqrt[q]{1 - q\beta_2 + c_q\theta_2^q} + \frac{\sigma_2^{q-1}}{\gamma_2 - \lambda_2 m_2}\end{aligned}$$

$$\sqrt[q]{\tau_2^q\theta_2^q - q\lambda_2 r_2 + c_q\lambda_2{}^q s_2^q} + \frac{l_{22}\lambda_2\sigma_2^{q-1}}{\gamma_2 - \lambda_2 m_2} + \sum_{k\in\Gamma,k\neq 2}\frac{\lambda_k\sigma_k^{q-1}}{\gamma_k - \lambda_k m_k}(t_{k2} + l_{k2}),$$

$$\ldots,$$

$$\sqrt[q]{1 - q\beta_p + c_q\theta_p^q} + \frac{\sigma_p^{q-1}}{\gamma_p - \lambda_p m_p}\sqrt[q]{\tau_p^q\theta_p^q - q\lambda_p r_p + c_q\lambda_p{}^q s_p^q} + \frac{l_{pp}\lambda_p\sigma_p^{q-1}}{\gamma_p - \lambda_p m_p}$$

$$\left.+\sum_{k=1}^{p-1}\frac{\sigma_k^{q-1}\lambda_k}{\gamma_k - \lambda_k m_k}(t_{k,p} + l_{k,p})\right\}.$$

It follows from hypothesis (4.1) that $0 < \xi < 1$.

Let $a_n = \sum_{i=1}^{p}\|x_i^n - x_i\|_i, \xi_n = \xi + (1-\xi)\alpha_n$. Then, (5.11) can be rewritten as $a_{n+1} \leq \xi_n a_n, n = 0,1,2,\ldots$. By (5.2), we know that $\limsup_n \xi_n < 1$, it follows from Lemma 5.1 that

$$a_n = \sum_{i=1}^{p}\|x_i^n - x_i\|_i \text{ converges to } 0 \text{ as } n \longrightarrow \infty.$$

Therefore, $\{(x_1^n, x_2^n, \ldots, x_p^n)\}$ converges to the unique solution $(x_1, x_2, \ldots, x_p)$ of problem (3.1). This completes the proof.

Remark 5.1 If E is 2-uniformly smooth and there exist constants $\lambda_i > 0 (i = 1,2,\ldots,p)$ such that,

$$\sqrt{1 - 2\beta_1 + c_2\theta_1^2} + \frac{\sigma_1}{\gamma_1 - \lambda_1 m_1}\sqrt{\tau_1^2\theta_1^2 - 2\lambda_1 r_1 + c_2\lambda_1{}^2 s_1^2} + \frac{l_{11}\lambda_1\sigma_1}{\gamma_1 - \lambda_1 m_1}$$

$$+\sum_{k=2}^{p}\frac{\lambda_k\sigma_k}{\gamma_k - \lambda_k m_k}(t_{k1} + l_{k1}) < 1, \sqrt{1 - 2\beta_2 + c_2\theta_2^2} + \frac{\sigma_2}{\gamma_2 - \lambda_2 m_2}$$

$$\sqrt{\tau_2^2\theta_2^2 - 2\lambda_2 r_2 + c_2\lambda_2{}^2 s_2^2} + \frac{l_{22}\lambda_2\sigma_2}{\gamma_2 - \lambda_2 m_2} + \sum_{k\in\Gamma,k\neq 2}\frac{\lambda_k\sigma_k}{\gamma_k - \lambda_k m_k}(t_{k2} + l_{k2}) < 1,$$

$$\ldots,$$

$$\sqrt{1 - 2\beta_2 + c_2\theta_p^2} + \frac{\sigma_p}{\gamma_p - \lambda_p m_p}\sqrt{\tau_p^2\theta_p^2 - 2\lambda_p r_p + c_2\lambda_p{}^2 s_p^2} + \frac{l_{pp}\lambda_p\sigma_p}{\gamma_p - \lambda_p m_p}$$

$$+\sum_{k=1}^{p-1}\frac{\sigma_k\lambda_k}{\gamma_k - \lambda_k m_k}(t_{k,p} + l_{k,p}) < 1$$

then (4.1) holds. It is worth noting that the Hilbert space and L_P (or l_p) spaces $(2 \leq q \leq \infty)$ are 2-unifomly smooth Banach spaces.

Remark 5.2 Theorems 4.1 and 5.1 unifies, improves and extends those results in Refs. [1,2,9,11,20–29,34–36] in several aspects.

Remark 5.3 By the results in Sects. 4 and 5, it is easy to obtain the existence of solutions and the convergence results of iterative algorithms for the special cases of problem (3.1). And we omit them here.

Acknowledgements The authors would like to express their thanks to the referees for helpful suggestions. This research was supported by the National Natural Science Foundation of China (Grant No. 10471159), the Science and Technology Research Project of Chinese Ministry of Education (Grant No. 206123) and the Postdoctoral Science Foundation of China (Grant No. 2005038133).

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